LECTURE NOTES: 4-7 OPTIMIZATION (PART 1)

QUESTION 1: What does *optimization* mean?

- getting the best solution
- the maximum speed
- the minimum cost
- the most or least of some quantity

QUESTION 2: Where might you encounter the need for optimization or where have you already encounter this?

- business (max profit)
- biology (min/max population)
- physics (max height, max velocity)

QUESTION 3: Is there anything wrong with a student who finds $2 \cdot 3$ by explaining:

I find $2 \cdot 3$ by adding the two numbers then adding 1 to get 6.

since the student got the right answer?

The student's method won't work in general: $5 \cdot 3 \neq 5+3+1$,

so the student will get most problems wrong.

A CAREFUL LOOK AT THE GOALS OF THIS SECTION:

- Getting the "right" $\boxed{\text{answer}}$ is only one small part of a problem. You need to get the problem right for the right reasons. (See Question 3 above.)
- Optimization is (arguably) the most common way you'll see Calc I outside of a Calculus course.
- There are essentially no routine problems, so you can't memorize your way to correct answers.
- What you should focus on:
  - (a) strategies for getting started, setting up, and thinking through any optimization problem
  - (b) strategies for establishing correctness with certainty.
- Things you should be asking yourself: Why did my teacher insist that should be written? Does my explanation look like the teachers? My neighbors?

A MODEL PROBLEM: TWO WAYS Find two positive numbers whose sum is 110 and whose product is a maximum.

thinking: If I am not sure how to begin, I think of specific examples that illustrate the things I am asked about — in this case — #'s that sum to 110 and their products.

Ex's

$1+109=110$ product: $1 \cdot 109 = 109$

$2+108=110$ product: $2 \cdot 108 = 216$ ← better! larger product

$10+100=110$ product: $10 \cdot 100 = 1{,}000$ ← even better!

Set up the general problem:

- Let x, y be positive numbers such that $x+y=110$ ← Explicitly identify your variables
- maximize the product: $P = xy$ ← Explicitly identify what quantity is being optimized.
- Using $y = 110 - x$, we have $P(x) = x(110-x)$ ← write quantity as a function of $\boxed{1}$ variable.
- with domain $[0, 110]$ since neither x nor y can be negative or larger than 110. ← Identify the domain.

METHOD 1 Closed-Interval Method.

Since $P(x) = 110x - x^2$, $P'(x) = 110 - 2x$.

So critical pts: $x = 55$.

x	55	0	110
$P(x)$	3025	0	0

↑ largest value is maximum

ANSWER: The maximum product is 3025 and occurs when the two numbers are both 55. (That is, when $x=y=55$.)

actually answer the question.

Method 2 Unique Critical Point Method

Since $P(x) = 110x - x^2$, $P'(x) = 110 - 2x$.
So $P(x)$ has one critical point $x = 55$.
Apply the First Derivative Test:

55

$+ + + 0 - - -$ ← sign of P'

to show that $x = 55$ is a local minimum.

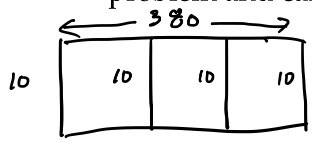
But, $P(x)$ is defined and continuous for all x-values in $[0, 110]$. Thus, the unique local extremum must be absolute.

[Now we draw the same conclusion]

PRACTICE PROBLEMS:

1. A rancher has 800 feet of fencing with which to enclose three adjacent rectangular corrals. What dimensions should be used so that the enclosed area will be a maximum?

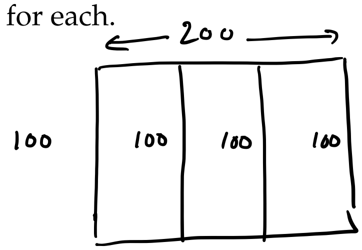
   (a) Draw and label with numbers two possible fencing arrangements of the type described in the problem and calculate the enclosed area for each.



$800-40=760$

$\frac{760}{2}=380$

area: $3800\ ft^2$



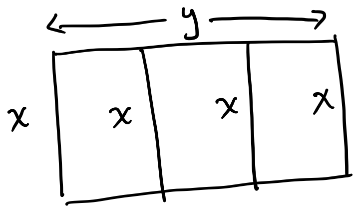
$800-400=400$

$\frac{400}{2}=200$

area: $20{,}000\ ft^2$

← better. more area.

   (b) Draw and label *with appropriate symbols* the general fencing arrangement.



$4x+2y=800$

   (c) Write an expression for the total enclosed area using your choice of symbols. Why are you asked to write an expression for *area* and not something else like perimeter or length or volume, etc?

$A = xy$

Why area? Because that is the quantity we are asked to optimize, specifically, maximize.

   (d) Write area as a function of *one* variable. Why is this step important? What is the *domain* of your function?

use $4x+2y=800$

or $y=400-2x$

to plug in

$A = xy = x(400-2x)$

<u>ANS</u>: $A(x) = 400x-2x^2$

domain: $[0, 200]$ since neither x nor y can be negative.

   (e) Finish the problem by finding the maximum. Show your work in an organized fashion, clearly justifying each step.

closed interval method:

$A'(x) = 400-4x = 0$.

crit pt: $x=100$

x	0	200	100
$A(x)$	0	0	20,000

↖ largest is max

Well, that's a fluke that I chose the max.

<u>Answer</u>: To maximize area, choose the partitions to have length 100 ft and the remaining side to have length 200 ft.

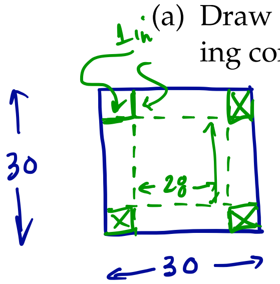
   (f) Is your answer reasonable? Explain.

It does seem like getting closer to a square is good.

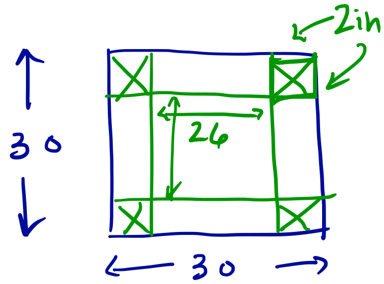
I could try some other x-value to check.

2. An open box of maximum volume is to be made from a square piece of material, 30 inches on a side, by cutting equal squares from the corners and turning up the sides. How should you cut out the corners so that the box has maximum volume?

   (a) Draw and label with numbers two possible choices of cut-out squares and volumes of resulting constructed boxes.



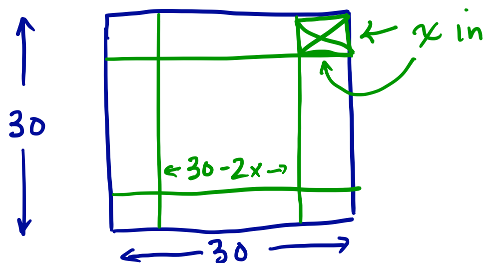
$V = l \cdot w \cdot h$
$= (28)(28)(1)$
$= 784 \text{ in}^2$



$V = l \cdot w \cdot h$
$= (26)(26)(2)$
$= 1352 \text{ in}^2$

better! more volume

   (b) Draw and label *with appropriate symbols* the general cut-out situation.



   (c) Write an expression for the total enclosed ___volume___ using your choice of symbols.

$$V = l \cdot w \cdot h = (30-2x)^2 \cdot x$$

   (d) Write ___Volume___ as a function of *one* variable. What is the *domain* of your function?

$V(x) = x(30-2x)^2$ ← already a function of 1 variable.

domain: $[0, 15]$

   (e) Finish the problem by finding the maximum. Show your work in an organized fashion, clearly justifying each step.

$V'(x) = (30-2x)^2 + x \cdot 2(30-2x)(-2)$
$= (30-2x)(30-2x-4x)$
$= (30-2x)(30-6x)$

crit pts: $x = 15, 5$

Closed Interval Method

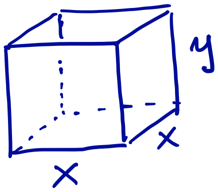
x	0	15	5
V	0	0	$20 \cdot 20 \cdot 5 = 2{,}000$ ← largest is maximum

answer: In order to maximize volume, 5 in squares should be cut out of each corner.

   (f) Is your answer reasonable? Explain.

The box would be $5 \times 10 \times 10$. Since there is no top, you would definitely want bottom longer than sides.

3. A manufacturer wants to design an open box having a square base and a surface area of 108 square inches. What dimensions will produce maximum volume?



surface area $= S = x^2 + 4xy = 108 \text{ in}^2$

Volume $= V = x^2 y$

Since we want to maximize volume, we write this as a function of one variable.

using S, we get: $y = \dfrac{108 - x^2}{4x} = 27x^{-1} - \frac{1}{4}x$

Substitute for y into V to get:

$V(x) = x^2\left(27x^{-1} - \frac{1}{4}x\right) = 27x - \frac{1}{4}x^3$.

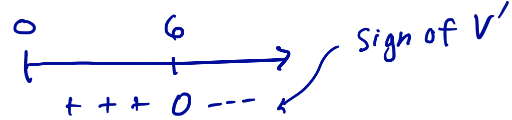
domain $(0, \infty)$ since length can't be negative.

First Derivative Test

$V'(x) = 27 - \frac{3}{4}x^2 = 0$

So $x^2 = 36$. So $x = 6$

($x = -6$ not in domain.)

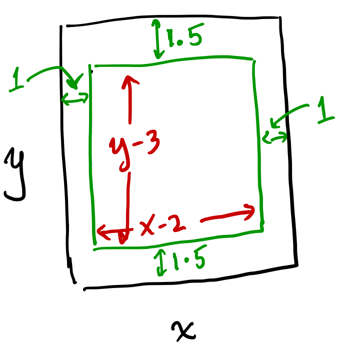


So V has a local maximum at $x=6$. Since this is the only critical point in the domain, it must correspond to an absolute maximum.

ANSWER To maximize volume, the base should be 6 in by 6 in and the height should be 3 in.

Note: One could use the Second Der. Test for Local Extrema, too.

4. A rectangular page is to contain 24 square inches of print. The margins at the top and bottom are to be 1.5 inches, and the margins on the left and right are to be 1 inch. What should the dimensions of the page be so that the least amount of paper is used?



Printed area $= (x-2)(y-3) = 24 \text{ in}^2$; area of paper $= P = xy$, minimize P

Solve for y: $y = \dfrac{24}{x-2} + 3$

Plug into P: $P(x) = x\left(\dfrac{24}{x-2} + 3\right) = 24x(x-2)^{-1} + 3x$, domain $(2, \infty)$

since length of paper needs space for 2 inches of margin.

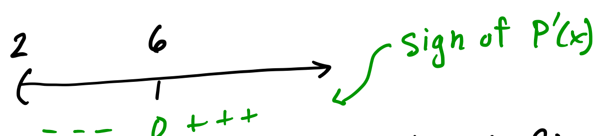
First Derivative Test:

$$P'(x) = 24(x-2)^{-1} + 24x(-1)(x-2)^{-2} + 3$$

$$= \frac{24}{x-2} - \frac{24x}{(x-2)^2} + 3 = \frac{24(x-2) - 24x + 3(x-2)^2}{(x-2)^2}$$

$$= \frac{3(x-6)(x+2)}{(x-2)^2}$$

$x=6$ is the only critical point in $(2, \infty)$

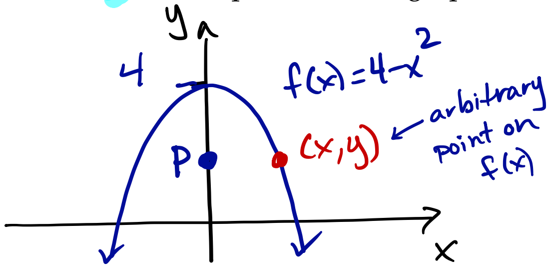


So $P(x)$ has a local min at $x=6$. Since it is unique, it is an absolute minimum.

ANSWER: To minimize page area, the base should be 6 inches and the height should be 9 inches.

Hint: minimize distance squared!!

5. Which points on the graph of $y = 4 - x^2$ are closest to the point $(0, 2)$?



distance from P to (x,y) $= \sqrt{x^2+(y-2)^2}$, minimize this.

Note: The square root gives me the heebie-jeebies.

So: minimize this instead: $D = x^2 + (y-2)^2$

Substitute in: $y = 4 - x^2$

$$D(x) = x^2 + (4-x^2-2)^2$$
$$= x^2 + (2-x^2)^2$$

domain: $[0, \infty)$

using symmetry, we know we'll need to pick the "mirror image" of our answer.

First Derivative Test

$$D'(x) = 2x + 2(2-x^2)(-2x) = 2x(2x^2-3)$$

critical points in domain: $x=2$, $x = \sqrt{3/2}$.

$$\begin{array}{ccc} 0 & \sqrt{3/2} & \\ [\!-\!-\!-\!-\!-\!-\!-\!-\!-\!-\!+\!-\!-\!-\!-\!-\!-\!-\!-\!-\!-\!\rightarrow & & \\ 0\ - - - & 0\ + + + & \end{array}$$

sign of $D'(x)$

So D has an absolute minimum at $x = \sqrt{3/2}$ and $x = -\sqrt{3/2}$.

ANSWER: The points on $y = 4-x^2$ closest to $(0,2)$ are $\left(\sqrt{3/2}, 5/2\right)$ and $\left(-\sqrt{3/2}, 5/2\right)$.